7. Probabilistic approach

7.1 Introduction

The aim of this chapter is to provide a deeper understanding of the principles and fundamentals of verification.

How, on the basis of the verification formats introduced earlier, can sufficient structural safety be defined, taking into account *the uncertainties associated with the various influencing parameters* (Fig. 7.1)?

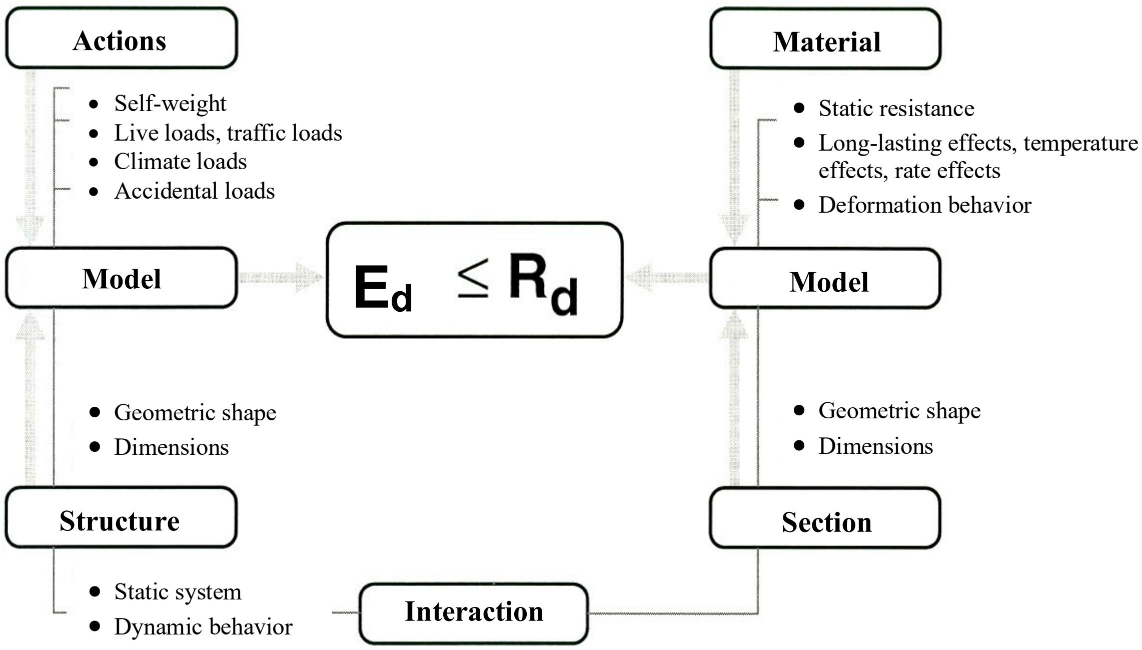


Figure 7.1 Uncertainties associated with influencing parameters.

Statistics and probability: Reminder of some *basic concepts*:
- histogram (Fig. 7.2),
- probability density function PDF (Fig. 7.3a), with its characteristics: the mean value (or 1st moment) and standard deviation (or 2nd moment), as well as its cumulative distribution function CDF (Fig. 7.3b), with the median value,
- Gauss curve or normal distribution, for which the mean and median values are identical, and is often expressed in its reduced centered form (Fig. 7.4).

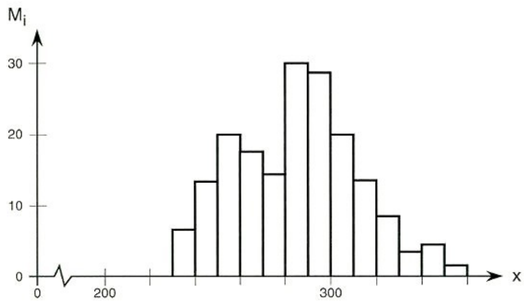


Figure 7.2 Histogram of a variable.

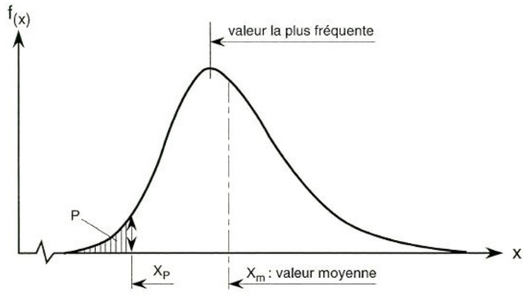


Figure 7.3a Probability density.

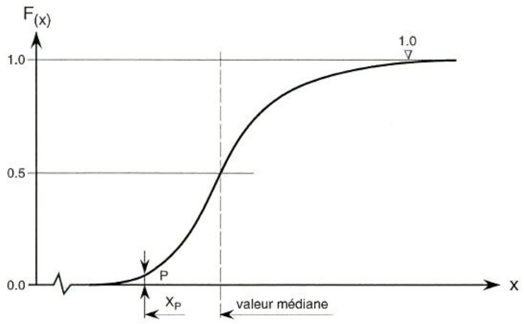


Figure 7.3b Distribution function.

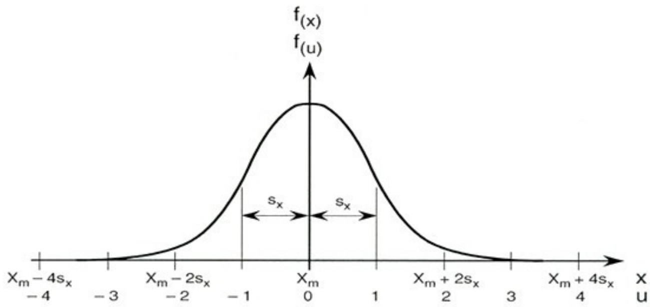


Figure 7.4 Gauss curve (reduced centered).

Normal distribution:

More on the normal distribution, with the definition and corresponding expression for the PDF and CDF below (Fig. 7.5a). It is useful to also give the values of the PDF and CDF in tabular form (Fig. 7.5b). Note in this table the values underlined in red, which correspond to:

- Mean plus one standard deviation, which correspond approx. to a probability of 16% (or the inverse, 84% of the data if one-sided, or 70% if two-sided)
- Mean plus 3.7 standard deviation, which correspond approx. to the upper bound of acceptable annual failure probability for structures, i.e. $1{\cdot}10^{-4}$ (more precisely 3.72 std dev.)
- Mean plus 4.7 standard deviation, which correspond approx. to the lower bound of acceptable annual failure probability for structures, i.e. $1{\cdot}10^{-6}$ (more precisely 4.75 std dev.)

Definition

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}u^2}$$

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}u^2}\, du$$

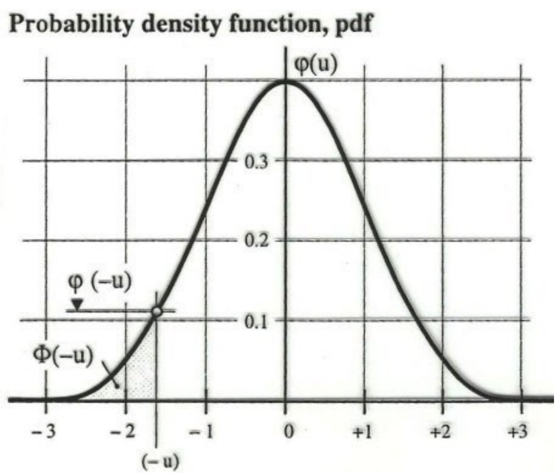


Figure 7.5a Normal distribution expression and plot of the PDF

u	Φ(– u)	φ(u)
.0	.500 00	.398 94
.1	.460 17	.396 95
.2	.420 74	.391 04
.3	.382 09	.381 39
.4	.344 58	.368 27
.5	.308 54	.352 07
.6	.274 25	.333 22
.7	.241 96	.312 25
.8	.211 86	.289 69
.9	.184 06	.266 09
1.0	.158 66	.241 97
1.1	.135 67	.217 85
1.2	.115 07	.194 19
1.3	.096 80	.171 37
1.4	.080 76	.149 73
1.5	.066 81	.129 52
1.6	.054 80	.110 92
1.7	.044 57	.094 05
1.8	.035 93	.078 95
1.9	.028 72	.065 62
2.0	.022 75	.053 99
2.1	.017 86	.043 98
2.2	.013 90	.035 48
2.3	.010 72	.028 33
2.4	.008 20	.022 40
2.5	.006 21	.017 53
2.6	.004 661	.013 58
2.7	.003 467	.010 42
2.8	.002 555	.007 92
2.9	.001 866	.005 95
3.0	.001 499	.004 43
3.1	.000 968	.003 27
3.2	.000 687	.002 38
3.3	.000 483	.001 72
3.4	.000 337	.001 23
3.5	.000 233	.008 73
3.6	.000 159 1	.000 612
3.7	.000 107 8	.000 425
3.8	.000 072 3	.000 292
3.9	.000 048 1	.000 199
4.0	.000 031 7	.000 134
4.1	.000 020 7	.000 089 3
4.2	.000 013 3	.000 058 9
4.3	.000 008 5	.000 038 5
4.4	.000 005 4	.000 024 9
4.5	.000 003 4	.000 016 0
4.6	.000 002 1	.000 010 1
4.7	.000 001 3	.000 006 4
4.8	.000 000 8	.000 003 9
4.9	.000 000 5	.000 002 4

Figure 7.5b Normal distribution table of values of the PDF and CDF

7.2 Limit state function and reliability index

The basic concepts of statistics and probability can be applied to a probabilistic analysis of structural safety (Fig. 7.6). They allow the concept of safety to be expressed using (Fig. 7.7):

- a *limit state function*, or failure function, G : G = R - S (resistance R, sollicitation or action effects S)
- and the *reliability index β,* which can be determined using the failure probability p_f.

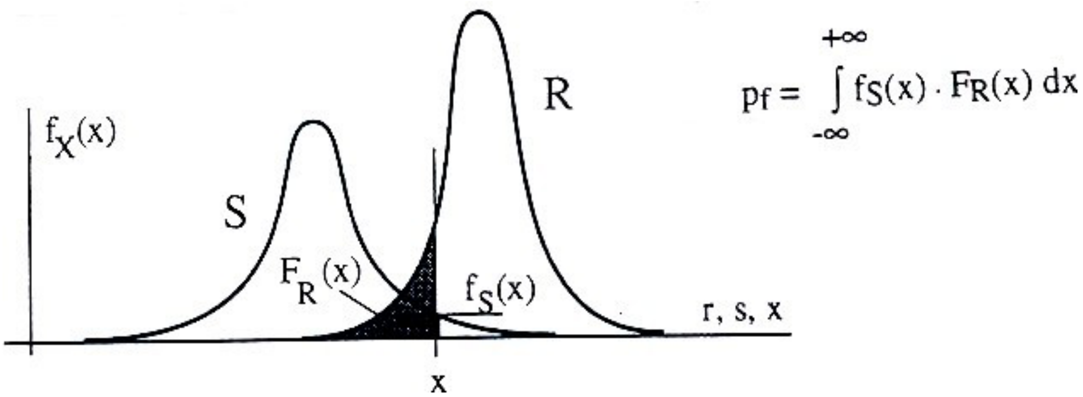


Figure 7.6 Probability density of resistance variables R and action effects variables S and determination of the probability that R is smaller than S.

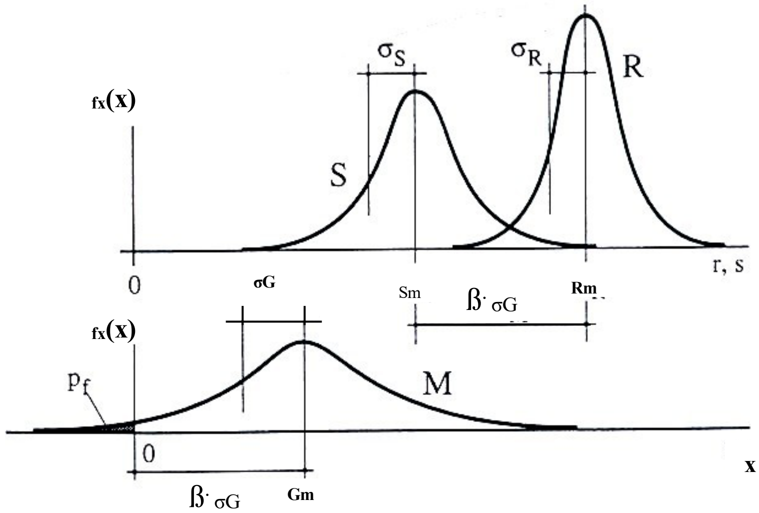


Figure 7.7 Probability density of the limit state function *G,* the resistance *R* and action effects *S* (according to Ernst Basler).

The *probability of failure* can be expressed as follows:

$$p_f = \int_{-\infty}^{0} f(G)dG = \Phi(-\beta)$$

Safety with respect to a limit state function G can be represented either by the failure probability p_f or by, in a normalized space, the reliability index β.

The reliability index represents the probability of failure, and there is a relationship between the two values. Figure 7.8 shows the relationship between the reliability index and the probability of failure, for a linear limit function and base variables following a normal distribution: $p_f = 1\text{-}\Phi(\beta)$.

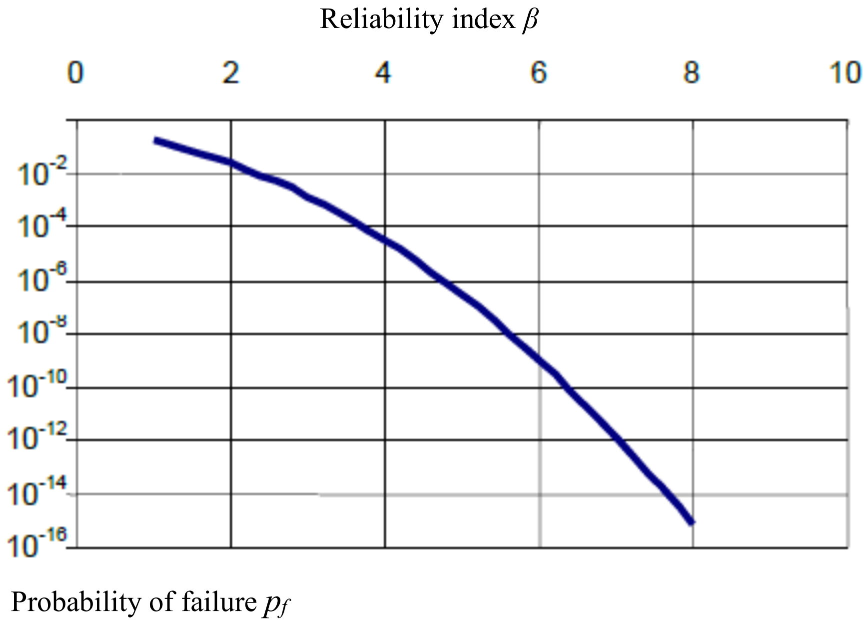


Figure 7.8 Relationship between reliability index and probability of failure

The important relationships that can be drawn from Figure 7.7 are as follows:

$$\beta = \frac{G_m}{\sigma_G}$$

With:
- G_m: mean value of the limit state function
- σ_G: standard deviation of the limit state function

In the case where the variables correspond to *normal distributions*:

$$G_m = R_m - S_m$$

$$\sigma_G = \sqrt{\sigma_R^2 + \sigma_S^2}$$

If the basic variables follow normal distributions and are uncorrelated, the reliability index can be calculated using the following formula (according to Hasofer):

$$\beta = \frac{R_m - S_m}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

This formulation is known as the “First Order Second Moment (FOSM) method”, which was introduced in 1974 by Hasofer. For a given limit function, the FOSM analysis can be used to find the mean value of the limit state function and the reliability index.

The FOSM method quickly becomes fairly complex once the number of variables to be considered exceeds two, and one variable does not follow a normal distribution. The FOSM method was subsequently extended and adapted to allow more general probabilistic treatment of variables. These include the Hasofer-Lind method, the linearization method and numerical calculation using Monte-Carlo simulation.

7.3 Verification of structural safety

According to the probabilistic approach, structural safety is verified when the following condition is satisfied:

$$\beta \geq \beta_{tgt} \quad (or\ p_f \leq p_{tgt})$$

With:

- β: reliability index
- p_f: Probability of ruin
- β_{tgt}: target reliability index
- p_{tgt}: target probability of failure

The reliability (or probability of failure) of an element (or system) is thus compared with a target or limit value. This target value expresses a certain *minimum reliability* of structures that is "expected by the public" (without taking into account economic optimization) and reflects society's requirements in terms of structural safety.

The minimum reliability can be determined by studying:

- structural accidents that have occurred (e.g. ruins, collapses, failures)
- the risk associated with the different activities of people (structure users)

In construction standards (design and dimensioning of new structures), a target reliability of 4.7 (or a probability of failure of 10^{-6} per year) is often assumed, for example, when "calibrating" partial safety factors (see chapters 4.4 and 4.5).

In the field of existing structures, an approach for establishing the target reliability index starts from the *specific hazard situation* and includes a risk analysis (risk = probability of occurrence x damage) using the following approach:

- The risk involved in a hazard situation is established by estimating the amount of damage likely to result from ruin and the value of use (and, where applicable, the immaterial values) of the structure.
- *Risk categories* can be defined and "calibrated" in relation to a study involving an analysis of structural accidents that have occurred and a comparison with other risks to which the individual is exposed (Table 7.1).
- The value of the target reliability index can be obtained as a function of the risk category determined for a specific hazard situation.

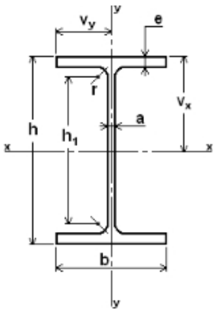
Risk category	Target probability (annual) p_{tgt}	Target reliability (annual) β_{tgt}
I	10^{-3}	3.1
II	10^{-4}	3.7
III	10^{-5}	4.2
IV	10^{-6}	4.7

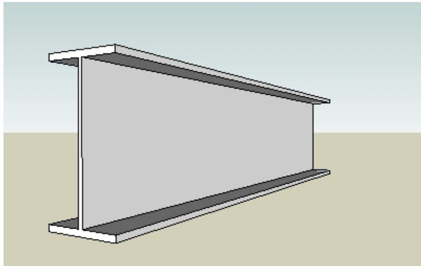
Table 7.1 Target probabilities and reliabilities by risk category

This approach can also be applied to verification of serviceability and fatigue (taking into account the probability of fatigue damage being detected by targeted monitoring).

<u>*Example:*</u>

Consider a simple beam of span L = 6 m in the form of a type IPE section in steel S235, subjected to a distributed load q (it is assumed that the self-weight of the section is neglected).





The beam is designed using a probabilistic approach with the following assumptions:

- target reliability index $\beta_0 = 4.0$
- $W_{y,pl}$ (moment of resistance of the section): constant value
- L (beam span): constant value
- f_y: normal distribution, with :
  - average value: $f_{ym} = 280$ N/mm^2
  - standard deviation: $\sigma_{fy} = 22.4$ N/mm^2
- q: normal distribution, with:

average value: $q_m = 0.9 q_k = 38.7$ kN/m’
standard deviation: $\sigma_q = 0.2 q_m = 7.74$ kN/m’

The limit function is G = R - S ≥ 0 where:

$$R = M_{pl} = f_y \cdot W_{y,pl}$$
$$S = \frac{qL^2}{8}$$

$W_{y,pl}$ can be determined by posing :

$$\beta = \frac{R_m - S_m}{\sqrt{\sigma_R^2 + \sigma_S^2}} \geq \beta_0 = 4.0 \left(which\ correspond\ at\ a\ probability\ of\ failure\ of\ p_f \approx 3 \cdot 10^{-5} \right)$$

The resistance is defined by:

$$R_m = M_{pl} = f_{ym} \cdot W_{y,pl} = 280 \cdot W_{y,pl}$$
$$\sigma_R = \sigma_{fy} \cdot W_{y,pl} = 22.4 \cdot W_{y,pl}$$

The loads are given by:

$$S_m = \frac{q_m L^2}{8} = 174.2 \cdot 10^6\ Nmm$$
$$\sigma_S = \frac{\sigma_q L^2}{8} = 34.8 \cdot 10^6\ Nmm$$

By integrating these values in:

$$R_m - S_m \geq 4 \cdot \sqrt{\sigma_R^2 + \sigma_S^2}$$

We find that:

$$W_{y,pl} \geq 1'262 \cdot 10^3\ mm^3 \quad \rightarrow \quad Choice\text{: } \boldsymbol{IPE\ 400}\ \ (W_{y,pl} \geq 1'310 \cdot 10^3\ mm^3)$$

However, the exact reliability index can be determined using the IPE 400 profile.

$$R_m = f_{ym} \cdot W_{y,pl} = 367\ kNm \quad and \quad \sigma_R = \sigma_{fy} \cdot W_{y,pl} = 29.3\ kNm$$
$$S_m = \frac{q_m L^2}{8} = 174.2\ kNm \quad and \quad \sigma_S = \frac{\sigma_q L^2}{8} = 34.8\ kNm$$

Therefore:

$$\beta = \frac{R_m - S_m}{\sqrt{\sigma_R^2 + \sigma_S^2}} = \frac{367 - 174.2}{\sqrt{29.3^2 + 34.8^2}} = 4.24$$

7.4 Interrelationship between probabilistic and deterministic approaches of structural safety

The equation for the reliability index (for uncorrelated base variables that follow a normal distribution) can be transformed by separating the terms for R and S:

$$\beta = \frac{R_m - S_m}{\sigma_G}$$

$$\frac{\sigma_R^2 + \sigma_S^2}{\sigma_G} \cdot \beta = R_m - S_m$$

$$S_m \cdot \left(1 + \beta \frac{\sigma_S}{S_m}\frac{\sigma_S}{\sigma_G}\right) = R_m \cdot \left(1 - \beta \frac{\sigma_R}{R_m}\frac{\sigma_R}{\sigma_G}\right)$$

With:

- $\beta = \beta_0$: target value of the reliability index
- α_S: influence factor of S ($= \sigma_S/\sigma_G$)
- α_R: influence factor of R ($= \sigma_R/\sigma_G$)
- ν_S: coefficient of variation of S ($= \sigma_S/S_m$)
- ν_R: coefficient of variation of R ($= \sigma_R/R_m$)

We obtain the following expressions for the examination values:

$$S^* = S_m(1 + \beta_0 \alpha_S \nu_S)$$
$$R^* = R_m(1 - \beta_0 \alpha_R \nu_R)$$

This gives the relationship between the examination value according to the probabilistic approach and the examination value according to the deterministic approach (expressed by a characteristic value and a partial factor):

$$X^* = \gamma_X \cdot X_k$$

With:

- X^*: examination value (S^* or R^*)
- γ_X: partial factor (γ_Q: load factor, $1/\gamma_M$: resistance factor (coefficient))
- X_k: characteristic value of a base variable (Q_k: characteristic value of a variable load, R_k: characteristic resistance value)

Hence the partial factor can be determined as follows:

$$\gamma_Q = \frac{S_m}{Q_k}(1 + \beta_0 \alpha_S \nu_S)$$

$$\gamma_M = \frac{R_k}{R^*} = \frac{R_k}{R_m(1 - \beta_0 \alpha_R \nu_R)}$$

Note: This formulation is used to determine partial factors defined in standards. Mean values, coefficients of variation, coefficients of influence and the value of the target reliability index are the key parameters for calibrating partial factors and representative values defined in design standards. A partial factor is therefore a function of the importance of a base variable in a limit function, the target reliability index and the uncertainty associated with the base variable.

The general expressions for the *verification of structural safety*, with the results of a probabilistic approach, are:

- probabilistic analysis: **S* ≤ R***
- deterministic approach (standards format): $\mathbf{E_d \leq R_d}$ (in the past: $S_d \leq R_d$)

A probabilistic approach to safety can be applied to verification using the standard's verification format (Fig. 7.9):

- the average value of a variable,
- a probabilistic value corresponding to a certain fractile,
- or a representative value.

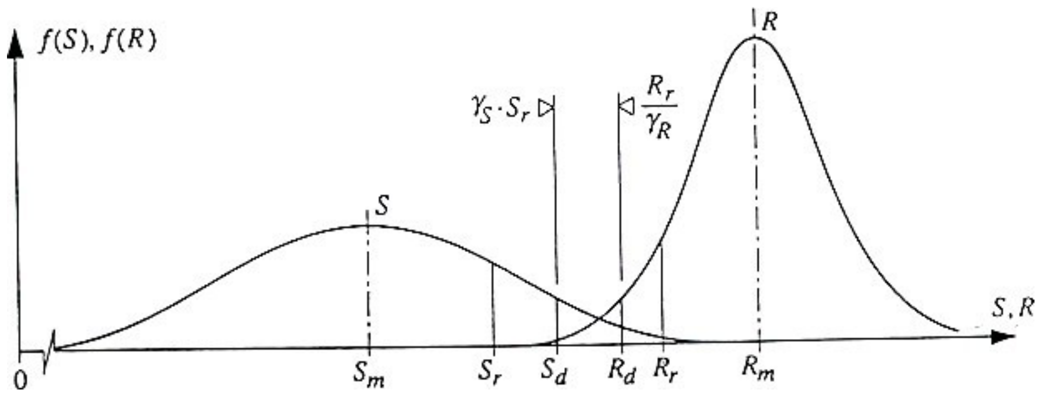


Figure 7.9 Application to verification

7.5 Probabilistic modelling of an action

7.5.1. Characteristic value of dynamic wind pressure

Climatic actions are sometimes modelled on the basis of meteorological measurement data and using a probabilistic analysis. The climatic action is thus associated with a certain *probability of occurrence* and a *certain return period*.

The characteristic value of climate action can then be based on two approaches:

I. Or on an examination value Q_d with a probability of not being exceeded p_T during fixed duration of use: $Q_d = Q(p_T)$
II. Or on a characteristic value Q_k corresponding to a certain return period. It gives: $Q_d = \gamma_Q \cdot Q_k$

Wind speed values can be assumed to follow a Gumbel-type I distribution law (extreme value I max.) with the distribution function (Fig. 7.10):

$$F(x) = e^{-e^{-\alpha(x-\mu)}}$$

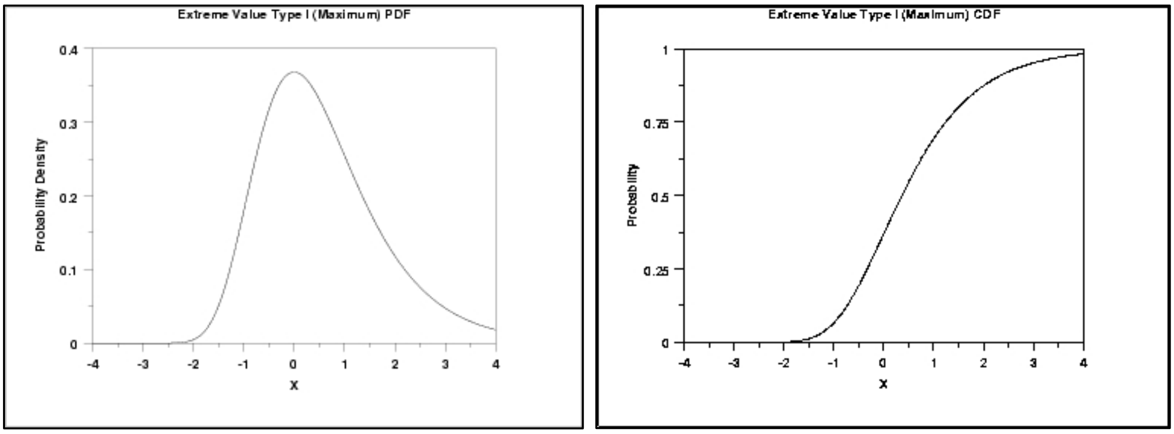
where:

- x: variable (wind speed)
- Mean value:

$$\mu = u + \frac{\gamma}{\alpha} \quad with \quad \gamma \cong 0.577216$$

- Standard deviation:

$$\sigma = \frac{\pi}{\alpha\sqrt{6}}$$



probability density distribution function

Figure. 7.10 Gumbel distribution I

Both approaches are illustrated using a numerical example.

The wind pressure acting on a tower to be designed must be determined on the basis of statistical values from a meteorological measuring station, with the following results relating to maximum and annual wind speed:

- mean value: μ = 40 m/s
- standard deviation: σ = 6 m/s

Approach I:

The dynamic wind pressure $q_d = q(p_T)$ is determined by considering the probability of non-overtaking $F(x_T)$. The statistical values of wind speed follow a Gumbel I distribution.

The parameters of the distribution function are determined as follows:

$$\alpha = \frac{\pi}{\sigma\sqrt{6}} = \frac{\pi}{6\sqrt{6}} = 0.2137\ s/m$$
$$u = \mu - \frac{\gamma}{\alpha} = 40 - \frac{0.577216}{0.2137} = 37.7\ m/s$$

The distribution function is then transformed to obtain X_T:

$$X_T = u - \frac{\ln\left(-\ln\left(F(X_T)\right)\right)}{\alpha}$$

The wind speed corresponds to a probability of not exceeding, for example $F(X_T) = 0.995$, therefore:

$$X_T = 37.7 - \frac{-5.29582}{0.2137} = 62.1\ m/s$$

And the dynamic wind pressure is determined according to:

$$q_T = \frac{1}{2}\rho X_T^2$$

with ρ the air density (approx.. 1.2 kg/m^3).

The uncertainty in the modelling of the effects of actions is considered by the partial factor γ = 1.10. The *examination value* of the dynamic wind pressure is thus:

$$q_d = \gamma_S \cdot q_T = 1.10 \cdot \left(\frac{1}{2} \cdot 1.2 \cdot 62.1^2\right) = 2.54\ kN/m^2$$

Finally, the <u>characteristic value</u> q_k of the dynamic wind pressure is obtained:

$$q_k = \frac{q_d}{\gamma_Q} = \frac{2.54}{1.5} = 1.69\ kN/m^2$$

Approach II:

The characteristic value q_k corresponding to a certain return period is determined by applying Gumbel's law.

A return period of 50 years can be accepted (i.e. 2 events exceeding the characteristic value during a period of use of 100 years). This corresponds to a value of distribution function $F(X_k) = 0.98$.

By applying the same equations (as before with approach I), the characteristic value of the wind speed

X_k is calculated:

$$X_k(F = 0.98) = 55.5\ m/s$$

and the characteristic value q_k of the dynamic wind pressure is:

$$q_k = \frac{1}{2} \cdot 1.2 \cdot 55.5^2 = 1.85\ kN/m^2$$

Note: With these numerical values, approach II provides a characteristic value for dynamic wind pressure that is 9.5% higher than with approach I.

7.5.2 Examination value determined from an empirical distribution

Axle load measurements of road traffic are analyzed to determine the examination value on the basis of an empirical distribution.

Axle loads do not follow a specific distribution law with sufficient precision (Fig. 7.11).

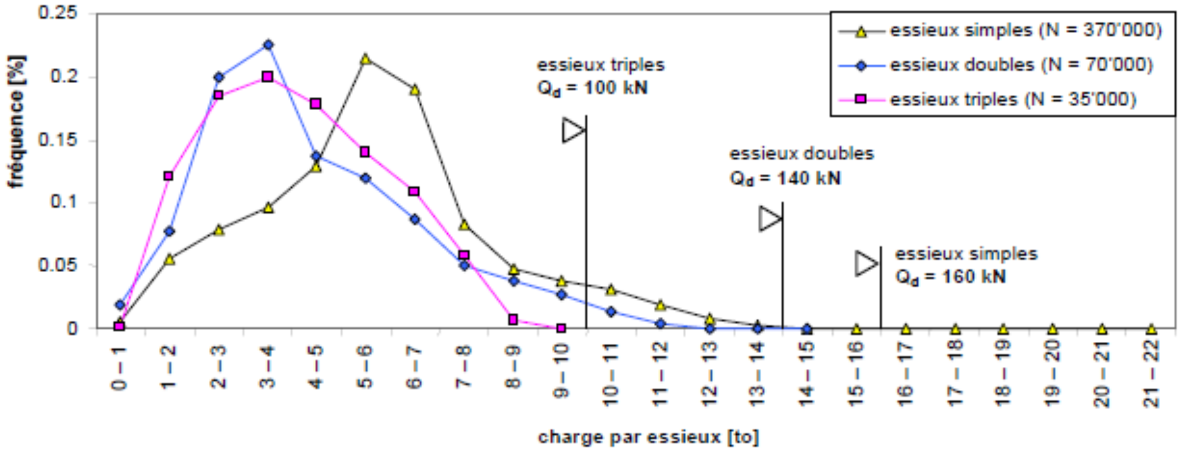


Figure. 7.11: Frequency of measured axle load as a function of axle type and statistical analysis [Ludescher 2004]. (These are traffic load measurements from 5 measuring stations on Swiss motorways for the periods May to September 2000 and March to July 2001).

The characteristic value can be determined by assuming that the probability of all measured axle load values remaining below the axle load examination value remains below the target probability for the leading action:

$$P(Q < Q^*) < \Phi\big(\beta = \beta_{target} \cdot \alpha_Q\big)$$

with:
- β_{target}: target reliability index (see Chap. 7.3)
- α_Q: influence factor of Q (a value of $\alpha_Q = 0.8$ is often used for leading action)
- Φ: function of the standard normal distribution

Note: leading (in French *prédominant*), and *accompanying* (in French *concomitant*)
With a reliability index $\beta_0 = 4.0$, we obtain:

$$P(Q < Q^*) < \Phi\big(\beta = \beta_0 \cdot \alpha_Q = 3.2\big) = 0.9993$$
$$or \quad P(Q > Q^*) 6.9 \cdot 10^{-4}$$

The examination value is then determined directly from the empirical distribution (results of a measurement campaign) for P = 0.9993.

The precision of the resulting value is obtained by applying the rules for the geometric distribution; the standard deviation of the probability of exceeding the characteristic value is thus:

$$\sigma = \frac{\sqrt{n}}{n}$$

with:

- $\bar{n}$: number of measurements exceeding the characteristic value
- n: total number of measurements (of axle loads)

The examination value Q_d for this road traffic action is finally obtained by taking into account also partial factor γ_S which takes account of uncertainties in modelling the effects of actions (see Chap. 4.5.1), i.e. actions (due to road traffic) that are not explicitly covered by the hazard situation (measurement results).

$$Q_d = \gamma_S \cdot Q^*$$

with:

- γ_S: model factor (often $\gamma_S = 1.10$)

Table 7.2 shows the Q_d determined (with the precision expressed by the standard deviation) for the measurement results given in Figure 7.11.

Table 7.2 Examination values for road axle loads

Single axles (370'424 measurements)	$Q^* \approx 145$ kN or $P(Q > 145 \text{ kN}) \approx 7.0 \cdot 10^{-4} \pm 0.4 \cdot 10^{-4}$ $\rightarrow Q_d = \gamma_S \cdot Q^* \approx$ **160 kN**
Twin axles (69'453 measurements)	$Q^* \approx 125$ kN or $P(Q > 125 \text{ kN}) \approx 5.5 \cdot 10^{-4} \pm 0.8 \cdot 10^{-4}$ $\rightarrow Q_d = \gamma_S \cdot Q^* \approx$ **140 kN**
Triple axles (34'662 measurements)	$Q^* \approx 90$ kN or $P(Q > 90 \text{ kN}) \approx 6.1 \cdot 10^{-4} \pm 1.3 \cdot 10^{-4}$ $\rightarrow Q_d = \gamma_S \cdot Q^* \approx$ **100 kN**

Note: The values in Table 7.2 concerns the static load. Moreover, a dynamic effect due to vehicle movement should be taken into account, which is often done by an amplificatory factor φ_i of the static load.

Application 1 – Test values for various action effects

The examination value of the effect of actions E_d (verification of structural safety) and the service value of the effects of actions E_d (verification of serviceability) are to be determined for existing structures and for the following four examination situations:

- Example 1 - Building slab, live load as leading action
- Example 2 - Roof of an ice rink, snow or wind as the leading action
- Example 3 - Rail bridge, railway loads (normal track) as the leading action
- Example 4 - Impact against a column

Actions should be described using their correct abbreviations (acronyms).

Load factors and reduction coefficients should be described by their abbreviations (acronyms) and numerical values. Even if described with the index k, the characteristic values of the loads may be updated values.

Solution

Example 1 – Building slab (existing structure)

Examination situation: live load as leading action (limit state type 2)

1) Verification of structural safety

Examination value of the effect of actions:

$$E_d = E\{\gamma_G G_{k0}, \gamma_G G_{k1}, \gamma_{Q1} Q_{k1}(live\ load)\}$$

Permanent actions:
- γ_G: load factor for permanent action; limit state type 2, $\gamma_G = 1.20$ (unfavorable effect)
- G_{k0}: characteristic value of the self-weight of the structure
- G_{k1}: characteristic value of the imposed loads: screed, floor covering

Leading action:
- γ_{Q1}: load factor for leading (variable) action; $\gamma_{Q1,act} = 1.50$
- Q_{k1}(live load): characteristic value of the live load in the building (variable leading action)

Accompanying action(s): none

2) Verification of serviceability

a) Service value of the effect of irreversible actions, cases of *rare* actions (deformation due to the effects of shrinkage and creep of the reinforced concrete slab → "aptitude au fonctionnement"):

$$E_{ser} = E\{G_{k0}, G_{k1}, Q_{k1}(live\ load)\}$$

b) Service value of the effect of reversible actions, cases of *frequent* actions ((elastic) deflection due to live load → "aptitude au fonctionnement, comfort"):

$$E_{ser} = E\{G_{k0}, G_{k1}, \psi_{11} Q_{k1}(live\ load)\}$$

- ψ_{11}: reduction coefficient for the frequent value of variable action 1 ($\psi_{11} = 0.5$)

c) Service value of the effect of reversible actions, case of *quasi-permanent* actions → ("aspect"):

$$E_{ser} = E\{G_{k0}, G_{k1}, \psi_{21} Q_{k1}(live\ load)\}$$

- ψ_{21}: reduction coefficient for the quasi-permanent value of variable action 1 ($\psi_{21} = 0.3$)

Note: In general, case b) "Service value of the effect of reversible actions, case of frequent actions" is relevant to be analyzed by calculation. Cases a) and c) are verified by a survey of the current state of the slab. In the case of a slab to be built, the three cases a) to c) must be analyzed/verified.

Example 2 – Roof of an ice rink at an altitude of 1'500m (existing structure)

Examination situation 1: snow as the leading action (limit state type 2)

1) Verification of structural safety:

Examination value of the effect of actions:

$$E_d = E\left\{\gamma_G G_{k0}, \gamma_G G_{k1}, \gamma_{Q1} Q_{k1}(snow), \psi_0 Q_k(wind)\right\}$$

Permanent actions:
- γ_G: load factor for permanent action; limit state type 2, $\gamma_G = 1.20$ (unfavorable effect)
- G_{k0}: characteristic value of the self-weight of the structure
- G_{k1}: characteristic value of the imposed loads: roof covering and waterproofing, suspended elements

Leading action:
- γ_{Q1}: load factor for leading (variable) action; $\gamma_{Q1,act} = 1.50$
- Q_{k1}(snow): characteristic value of the snow load (variable leading action)

Accompanying action:
- $\psi_0 Q_k$(wind): rare value of wind (variable) action
- ψ_0: reduction coefficient for the rare value of a wind (variable) action ($\psi_0 = 0.6$)
- Q_k(wind): characteristic value of the wind load

2) Verification of serviceability

In general, case b) "Service value of the effect of reversible actions, case of frequent actions" is relevant to be analyzed and verified by calculation. Cases a) and c) are verified by a survey of the current state of the slab.

b) Service value of the effect of reversible actions, case of *frequent* actions ((elastic) deflection due to snow load ➔ "aptitude au fonctionnement, comfort"):

$$E_{ser} = E\{G_{k0}, G_{k1}, \psi_{11} Q_{k1}(snow), \psi_2 Q_k(wind)\}$$

- ψ_{11}: reduction coefficient for the frequent value of snow load ($\psi_{11} = 0.83$)
- $\psi_2 Q_k$(wind): quasi-permanent value of the wind load ($\psi_2 = 0$)

Examination situation 2: wind as the leading action (limit state type 2)

1) Verification of structural safety

Examination value of the effect of actions:

$$E_d = E\left\{\gamma_G G_{k0}, \gamma_G G_{k1}, \gamma_{Q1} Q_{k1}(wind), \psi_0 Q_k(snow)\right\}$$

Permanent actions:
- γ_G: load factor for permanent action; limit state type 2, $\gamma_G = 1.20$ (unfavorable effect)
- G_{k0}: characteristic value of the self-weight of the structure
- G_{k1}: characteristic value of the imposed loads: roof covering and waterproofing, suspended elements

Leading action:
- γ_{Q1}: load factor for leading (variable) action; $\gamma_{Q1,act} = 1.50$
- Q_{k1}(wind): characteristic value of the wind load (variable leading action)

Accompanying action:
- $\psi_0 Q_k$(snow): rare value of snow (variable) action
- ψ_0: reduction coefficient for the rare value of a snow (variable) action ($\psi_0 = 0.96$)
- Q_k(snow): characteristic value of the snow load

2) Verification of serviceability

In general, case b) "Service value of the effect of reversible actions, case of frequent actions" is relevant to be analyzed and verified by calculation. Cases a) and c) are verified by a survey of the current state of the slab.

b) Service value of the effect of reversible actions, case of *frequent* actions ((elastic) deflection due to wind load ➔ "aptitude au fonctionnement, comfort"):

$$E_{ser} = E\{G_{k0}, G_{k1}, \psi_{11} Q_{k1}(wind), \psi_2 Q_k(snow)\}$$

- ψ_{11}: reduction coefficient for the frequent value of wind load ($\psi_{11} = 0.5$)
- $\psi_2 Q_k$(snow): quasi-permanent value of the snow load ($\psi_2 = 0.33$)

Example 3 – Rail bridge (existing structure)

Examination situation: railway loads (normal track: SBB, BLS, etc.) as the leading action (limit state type 2)

1) Verification of structural safety:

Examination value of the effect of actions:

$$E_d = E\{\gamma_G G_{k0}, \gamma_G G_{k1}, \gamma_{Q1} Q_{k1}(SBB), \psi_0 Q_k(wind)\}$$

Permanent actions:

- γ_G: load factor for permanent action; limit state type 2, $\gamma_G = 1.20$ (unfavorable effect)
- G_{k0}: characteristic value of the self-weight of the structure
- G_{k1}: characteristic value of the imposed loads: equipment, rails, sleepers, ballast

Leading action:

- γ_{Q1}: load factor for leading (variable) action; $\gamma_{Q1} = 1.45$
- $Q_{k1}(SBB)$: characteristic value of the railway load (variable leading action)

Accompanying action:

- $\psi_0 Q_k(wind)$: rare value of wind (variable) action
- ψ_0: reduction coefficient for the rare value of a wind (variable) action ($\psi_0 = 0.8$)
- $Q_k(wind)$: characteristic value of the wind load

Tableau 11 : Coefficients de réduction pour les ponts-rails à voie normale

Actions	ψ_0	ψ_1	ψ_2
Charges verticales			
– Modèle de charge 1	1,0	1,0 1)	0 2)
– Modèle de charge 2	1,0	1,0 1)	0 2)
– Modèle de charge 3	0	1,0	0 3)
Forces horizontales 4)	1,0	1,0 1)	0
Forces dues au vent			
– En général	0,8	0,5	0
– Forces aérodynamiques dues au trafic ferroviaire	1,0	0,5	0
Effets de la température	0,6	0,6	0,5
Actions du terrain de fondation			
– Poussée des terres	0,7	0,7	0,7
– Pression hydraulique	0,7	0,7	0,7

2) Verification of serviceability

In general, case b) "Service value of the effect of reversible actions, case of frequent actions" is relevant to be analyzed and verified by calculation. Cases a) and c) are verified by a survey of the current state of the slab.

b) Service value of the effect of reversible actions, case of *frequent* actions ((elastic) deflection due to railway load ➔ “aptitude au fonctionnement, comfort”):

$$E_{ser} = E\{G_{k0}, G_{k1}, \psi_{11} Q_{k1}(SBB)\}$$

- ψ_{11}: reduction coefficient for the frequent value of railway load (ψ_{11} = 1.0)

Example 4 – Impact against a column (existing structure)

Examination situation: impact force as the leading action

1) Verification of structural safety

Examination value of the effect of accidental action (impact):

$$E_d = E\{G_{k0}, G_{k1}, A_d, \psi_{21} Q_k(live\ load)\}$$

- G_{k0}: characteristic value of the self-weight of the structure
- G_{k1}: characteristic value of the imposed loads: equipment, screed, floor covering
- A_d: shock force examination value
- $\psi_{21}Q_k$(live load): quasi-permanent value of live load (variable action) in combination with the accidental action (ψ_{21} = 0.6)

2) Verification of serviceability

Not required

Application 2 – Probabilistic verification of a bridge structure

The structure of an existing footbridge (simple beam with a span of 29.0m and a moment of inertia of $I_0 = 0.07254m^4$) is to be checked using the probabilistic approach.

- Check the structural safety of the mid span section (bending moment) and the cross-section close to the support (shear force) in compliance with a reliability index $\beta_0 = 4.0$ (level of required security).
- Check the serviceability in relation to the deflection of the simple beam, respecting a reliability index $\beta_0 = 3.3$.

Note: all the parameters can be modelled with a normal distribution.

Discuss the results.

Actions and characteristics of materials:

Actions	Type	Average value	Standard deviation
	Self-weight of load-bearing structure	$g_{0,m} = 13.0$ kN/m	$\sigma_{g0} = 0.50$ kN/m
	Imposed loads (cladding, equipment)	$g_{1,m} = 1.0$ kN/m	$\sigma_{g1} = 0.10$ kN/m
	Live load (pedestrians, cyclists)	$q_m = 12.0$ kN/m	$s_q = 1.0$ kN/m
Resistance of the supporting structure	The ultimate bending moment	$M_{R,m} = 4750$ kNm	$\sigma_M = 250$kNm
	Ultimate shear force	$V_{R,m} = 780$ kN	$\sigma_V = 45$ kN
	Modulus of elasticity	$E_m = 50.0$ kN/mm^2	$\sigma_E = 2.0$ kN/mm^2

Solution

The simple beam of the footbridge (29.0m span) is checked using the probabilistic approach.

Verification of structural safety: for the section at mid-span (bending moment) and the section close to the support (shear force) in compliance with a reliability index $\beta_0 = 4.0$ (required safety).

The limit function is $G = R - S \geq 0$ and the verification:

$$\beta = \frac{G_m}{\sigma_G} \geq \beta_0 = 4.0$$

With:

- For the bending moment at mid-span:

$$R = M_R$$
$$S = M_a = 0.125 \cdot l^2 \cdot \left(\sum g_i; q_i\right)$$

$$\beta = \frac{G_m}{\sigma_G} = \frac{M_{R,m} - M_{a,m}}{\sqrt{\sigma_{MR}^2 + \sigma_{Ma}^2}} \geq \beta_0 = 4.0$$

- For the shear force close to the support:

$$R = V_R$$
$$S = V_a = 0.5 \cdot l \cdot \left(\sum g_i; q_i\right)$$
$$\beta = \frac{G_m}{\sigma_G} = \frac{V_{Ru,m} - V_{a,m}}{\sqrt{\sigma_{VR}^2 + \sigma_{Va}^2}} \geq \beta_0 = 4.0$$

Bending moment at mid-span

$$R = M_R$$
$$S = M_a + M_{q,snow} = 0.125 \cdot l^2 \cdot \left(\sum g_i; q_i\right) + 0.125 \cdot l^2 \cdot \psi_0 \cdot q_{k,snow}$$

Notes:
- Snow load is not considered as an accompanying action, see Exercise 6, table with reduction coefficients, p.128.
- In principle, the design situation with snow load as the leading action should be analyzed. This design situation may be decisive if the footbridge is located at a high altitude with a correspondingly high snow load.

Action effects:

Mean value:

$$M_{a,m} = 0.125 \cdot l^2 \cdot (g_{0,m} + g_{1,m} + q_m)$$
$$M_{a,m} = 0.125 \cdot 29^2 \cdot (13.0 + 1.0 + 12.0) = 2733\ kNm$$

Standard deviation:

$$\sigma_{Ma} = 0.125 \cdot l^2 \cdot \sqrt{\sigma_{g0}^2 + \sigma_{g1}^2 + \sigma_q^2}$$
$$\sigma_{Ma} = 0.125 \cdot 29^2 \cdot \sqrt{0.50^2 + 0.1^2 + 1.0^2} = 0.125 \cdot 841 \cdot 1.122 = 118.0\ kNm$$

Resistance:

Mean value:
$$M_{R,m} = 4750\ kNm$$

Standard deviation:
$$\sigma_{MR} = 250.0\ kNm$$

Verification:

$$\beta = \frac{M_{R,m} - M_{a,m}}{\sqrt{\sigma_{MR}^2 + \sigma_{Ma}^2}} = \frac{4750 - 2733}{\sqrt{250^2 + 118^2}} = \frac{2017}{276.4} = 7.3 > \beta_0 = 4.0 \quad \rightarrow \quad OK$$

Shear force close to the support

Action effects:

Mean value:

$$V_{a,m} = 0.5 \cdot l \cdot \sum q_{m,i} = 0.5 \cdot 29 \cdot (13.0 + 1.0 + 12.0) = 377\ kN$$

Standard deviation:

$$\sigma_{Ma} = 0.5 \cdot l \cdot \sqrt{\sigma_{g0}^2 + \sigma_{g1}^2 + \sigma_q^2} = 0.5 \cdot 29 \cdot 1.122 = 16.3\ kN$$

Resistance:

Mean value:

$$V_{R,m} = 780.0\ kN$$

Standard deviation:

$$\sigma_{VR} = 45.0\ kN$$

Verification:

$$\beta = \frac{V_{R,m} - V_{a,m}}{\sqrt{\sigma_{VR}^2 + \sigma_{Va}^2}} = \frac{780 - 337}{\sqrt{45^2 + 16.3^2}} = \frac{403}{47.9} = 8.41 > \beta_0 = 4.0 \quad \rightarrow \quad OK$$

The structural safety (bending moment at mid-span and shear force close to the support) is satisfied.

Verification of the serviceability in relation to the deflection of the gangway, with a reliability index $\beta_0 = 3.3$. The serviceability criterion considered is users comfort. Consequently, by applying Table 9 given in the answers to Exercise 6, the deflection w_d is due only to variable actions (live load due to pedestrians and cyclists) and is calculated for the consequences of the effects of the actions reversible and the load case frequent.

$$C_d = \frac{l}{600} = \frac{29000}{600} = 48.3\ mm$$

The aim is to check that the stiffness of the beam is sufficient in relation to the variable action while complying with the maximum deflection limit criterion. Therefore:

$$C_d = \frac{l}{600} \geq w_d = \frac{5}{384} \cdot \frac{l^4}{EI} \cdot q$$

Hence, in terms of stiffness and action effects:

$$\frac{EI}{l^3} \geq 7.81 \cdot q$$

with the probabilistic variables E and q.

The limit function G = R – S is then:

$$G = \frac{EI}{l^3} - 7.81 \cdot q \geq 0$$

and the verification:

$$\beta = \frac{G_m}{\sigma_G} = \frac{\left(\frac{E_{cm} \cdot I}{l^3}\right)_m - 7.81 \cdot q_m}{\sqrt{\left(\frac{I}{l^3} \cdot \sigma_{Ec}\right)^2 + \left(7.81 \cdot \sigma_q\right)^2}} \geq \beta_0 = 3.3$$

with the moment of inertia of the mid-span section $I_0 = 0.07254\ m^4$:

$$\frac{E_{cm} \cdot I}{l^3} = \frac{50 \cdot 10^3\ N/mm^2 \cdot 72.54 \cdot 10^9\ mm^4}{29^3 \cdot 10^9\ mm^3} = 148.7\ \ N/mm$$

$$\sqrt{\left(\frac{I}{l^3} \cdot \sigma_{Ec}\right)^2 + \left(7.81 \cdot \sigma_q\right)^2} = \sqrt{\left(\frac{72.54 \cdot 10^9 \cdot 2.0 \cdot 10^3}{29^3 \cdot 10^9}\right)^2 + (7.81 \cdot 1.0)^2} = \sqrt{5.95^2 + 7.81^2} = 9.81\ \ N/mm$$

Verification:

$$\beta = \frac{148.7 - 7.81 \cdot 12.0}{9.81} = \frac{54.98}{9.81} = 5.60 > \beta_0 = 3.3 \quad \rightarrow \quad OK$$

The criterion for the serviceability is largely met.

Application 3 – Probabilistic determination of snow load

The roof of an existing ice rink needs to be checked. The skating rink is located in a village at an altitude of 2'150m in the Alps. The snow load to use during the verification must be determined more precisely using measurement results.

The snow load on a horizontal surface was measured at the skating rink over a period of 9 years, as shown in the table below.

With:
- m_i: number of measurements less than or equal to the value $S_{i,max}$
- T_i: return period

Ranking	$S_{i,max}$ [kN/m^2]	$p_i = m_i/(n+1)$ [-]	$X = -\ln(-\ln(p_i))$ [-]	$T_i = 1/(1-p_i)$ [years]
1 (2009)	11.5			
2 (2006)	11.8			
3 (2004)	12.7			
4 (2008)	13.0			
5 (2007)	13.8			
6 (2010)	14.0			
7 (2011)	14.1			
8 (2003)	14.2			
9 (2005)	15.0			

What is the value of the maximum snow load for a return period of 50 years during the future service life of 100 years, assuming that the maximum snow load values follow a Gumbel distribution?

The distribution function of the Gumbel distribution is:

$$F(x;\mu,\beta) = e^{-e^{(\mu-x)/\beta}}$$

The standard Gumbel distribution is obtained for $\mu = 0$ and $\beta = 1$:

$$F(x) = e^{-e^{-x}}$$

1. Determine the variable X according to the Gumbel's law (transformed into a linear scale) and enter the results in the table.
2. There is a linear relationship between the variable X and the snow load. Determine (graphically plotting the values on an X – $S_{i,max}$ diagram) by linear regression the parameters *a* and *b* of the equation:

$$S_{i,max}(p_w) \cong a \cdot X + b \; [kN/m^2]$$

3. Calculate the snow load for the 50-years return period: $S_{w,max}(T_w = 50$ years).

Solution

Each season, the value of the maximum snow load is determined for 9 seasons. The n = 9 results of the maximum snow load per season $S_{i,max}$ are reported and classified in the table below, with:

- m_i: number of values less than or equal to the $S_{i,max}$
- T_i: return period

For a return period of 50 years, the probability of occurrence is:

$$p_w = 1 - \frac{1}{T_w} \quad \rightarrow \quad p_w(T_w = 50\,years) = 0.98$$

1. The variable X is obtained by transforming the standard Gumbel distribution (distribution function) for μ = 0 and β = 1:

$$p_i(x) = e^{-e^{-x}} \quad \rightarrow \quad X = -ln\big(-ln(p_i)\big)$$

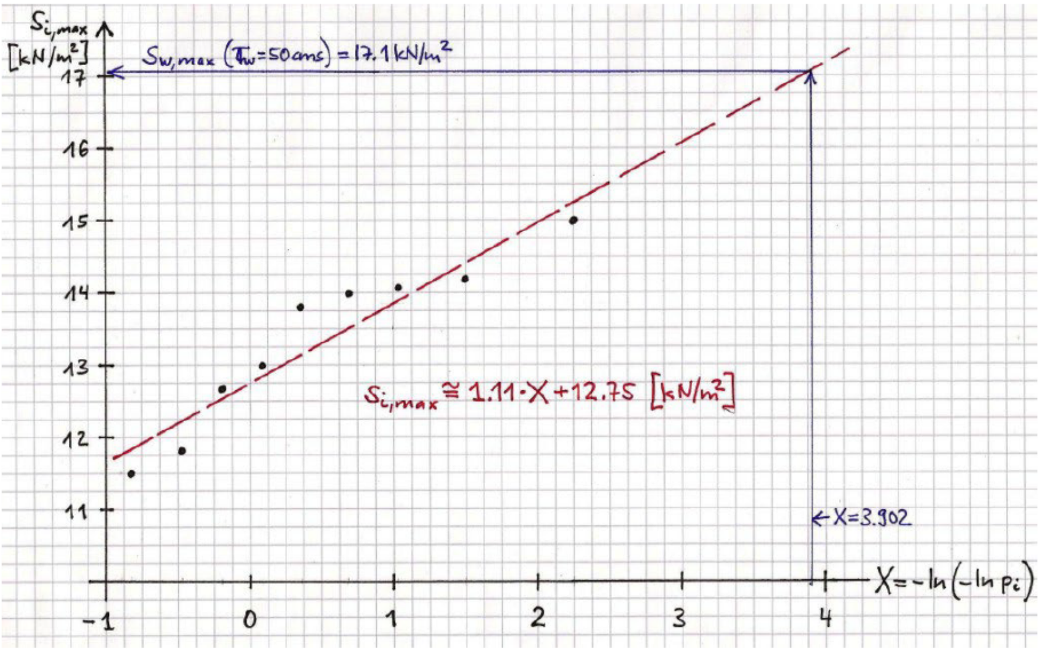
Note: the variable *X* is in fact a “translation” of the probability of occurrence on a scale for which the relationship between probability (X axis) and snow load (Y axis) follows a straight line.

The values of X_i are given in the table below for each probability of occurrence p_i.

Ranking	$S_{i,max}$ [kN/m²]	$p_i = m_i/(n+1)$ [-]	$X = -ln(-ln(p_i))$ [-]	$T_i = 1/(1-p_i)$ [years]
1 (2009)	11.5	0.1	-0.834	1.11
2 (2006)	11.8	0.2	-0.476	1.25
3 (2004)	12.7	0.3	-0.186	1.43
4 (2008)	13.0	0.4	0.088	1.67
5 (2007)	13.8	0.5	0.367	2.00
6 (2010)	14.0	0.6	0.671	2.50
7 (2011)	14.1	0.7	1.030	3.33
8 (2003)	14.2	0.8	1.500	5.00
9 (2005)	15.0	0.9	2.250	10.00
	17.1	0.98	**3.902**	50.00

2. Graphical determination (by linear regression) of the parameters *a* and *b* of the linear relationship between *X* and $S_{i,max}$, plotting the values on a $X - S_{i,max}$ diagram.

$$a \cong 1.11 \quad and \quad b = 12.75\ kN/m^2$$
$$S_{i,max}(p_i) \cong 1.11 \cdot X + 12.75\ [kN/m^2]$$



3. The snow load for the 50-years return period $S_{w,max}(T_w = 50 \text{ years})$ is for $p_w(T_w = 50 \text{ years}) = 0.98$:

$$S_{w,max} = 1.11 \cdot 3.902 + 12.75 = 17.1\ kN/m^2$$

$S_{w,max}(T_w = 50 \text{ years})$ is defined as the characteristic value S_k of the snow load (variable action) corresponding to a probability of occurrence of $p_w(T_w = 50 \text{ years}) = 0.98$.